A Conversation Between Constellations

"*Imagination will often carry us to worlds that never were, but without it we go nowhere*" - Carl Sagan

"But Dad, you told me that infinities don't exist... And now you're saying they do?"

Orion looked toward the sky for a few seconds, thinking of an intuitive answer.

"Well, it really depends who you ask, Lyra." he replied. "Ask a mathematician, and they will say that infinities certainly exist. In fact, mathematicians have even conjured ways to count infinities so that some infinities are bigger than others, if you can believe it."

Lyra's face contorted in disbelief and derision, but Orion carried on.

"Ask a physicist, however, and they will assure you that nature does not admit infinities, she instead prefers finite values which are measurable."

Lyra had become accustomed to being puzzled on her weekly walks with her father, but countable infinities was starting to push it.

"Why do mathematicians not have a problem with infinities?" asked Lyra, unwittingly trodding in an icy puddle as they strolled through the snowy park.

"Because it is the mathematician's job to deal in abstraction, and infinity is an abstract idea" explained Orion. "Think of a circle. Do circles really exist in nature, or do we simply approximate objects as being circular? Zoom in on any circle in the real world, and one will eventually see that circles are more like many-sided polygons. Mathematicians deal with perfect circles, not physical ones, and therefore operate in a realm of abstraction removed from our experiences."

"You mean, infinities do exist, but only in mathematics, not in the real world?" Lyra asked tentatively, now beginning to see the distinction.

"I believe so. This is how we can test if theories of physics are reasonable or not. If a theory predicts an infinity somewhere, something has clearly gone wrong." said Orion, turning now through a bright white field decorated with trees, pacing slowly and leaving deep footprints in the snow.

"So how do infinities happen in theories of physics then?" inquired Lyra.

"Good question. The most famous example is a black hole singularity"

"Black holes are scary." shuddered Lyra. "Even though we learnt about them in school, the thought still scares me."

Orion shot a look at his daughter as they both came to a standstill by an old oak tree jutted at the bottom of a large, gentle hill.

"And so they should. Black holes are regions of spacetime where the pull of gravity is so strong that nothing can escape, not even light itself."

Orion turned to Lyra.

"But that's not the real definition of course." he smiled.

Lyra was confused, having heard and seemingly understood that definition before.

"So what is?"

Orion picked up a large stick lying at the foot of the oak tree, shaking it free of snow and frost.

"The event horizon of a black hole is given by the boundary of the causal past of future null infinity within spacetime." [1]

"I'm sorry?" said Lyra, slightly annoyed and confused at all the jargon. Her father wryly smiled back, using the stick to trace two large boxes through the blanket of snow by the old tree. The boxes were angled at forty five degrees and attached in the middle, patched together with two triangles. It was an intriguing drawing.

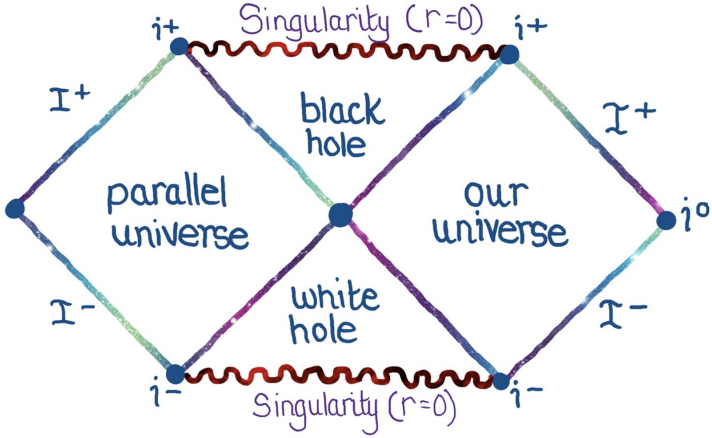

Figure 1: *Penrose diagram for Schwarzschild spacetime. $\mathcal{I}^+$ refers to future null infinity. $\mathcal{I}^-$ is past null infinity. The infinite future at a finite distance is denoted by i^+, with i^- being the time-reversal of i^+. Infinite distance but finite time is denoted by i^0* [2].

Orion pointed the stick to the box on the right.

"This is our universe. Everything that ever was, is or ever will be, is contained within the boundaries of this box" said Orion.

"Is there any reason why it's a box and not a circle or something?" puzzled Lyra.

"Good question. The boundaries of the box have very specific meanings, both mathematically and physically."

Orion indicated toward the upper right line of the box.

"This line represents the infinite future. More specifically, it is the infinite future for beams of light. If you imagine shining a torch into the sky, and then imagine following the light beam until of the time itself, and beyond, you will eventually be led to this line here. Formally, it's called future null infinity, because we use the word null to mean light beams." [2]

Orion examined Lyra as she looked intently at the diagram in the snow, a breeze blowing flakes of ice across the patch of universe below her. He continued.

"The line below it, in the bottom right, is called past null infinity which simply the same as future null infinity, but extrapolating backwards into the infinite past, instead of the infinite future."

Lyra had already spotted a problem.

"But the past isn't infinite, the Big Bang was billions of years ago yes, but not literally forever ago." she countered, confidently. Orion smiled again.

"Of course Lyra, you're right. But the objective of this diagram, called a Penrose diagram by the way, is to help us see causal structure within different types of spacetimes, rather than represent anything truly physical" he explained. [3]

"What do you mean by causal structure?" asked Lyra, uncertainly. This was all beginning to sound rather philosophical at this point.

Orion paused for a moment, leaning on his stick as though it were a staff, before once again drawing a shape in the snow. This time, however, it was what appeared to be two cones joined vertically at their vertices. It looked a little like an egg timer.

"You see where the tips of these cones meet?" Orion said. "That is the location of an event in spacetime, you may think of it as the present for whoever is at that location. This point could be you for instance, right now, standing beneath this oak tree at half past ten on a very cold

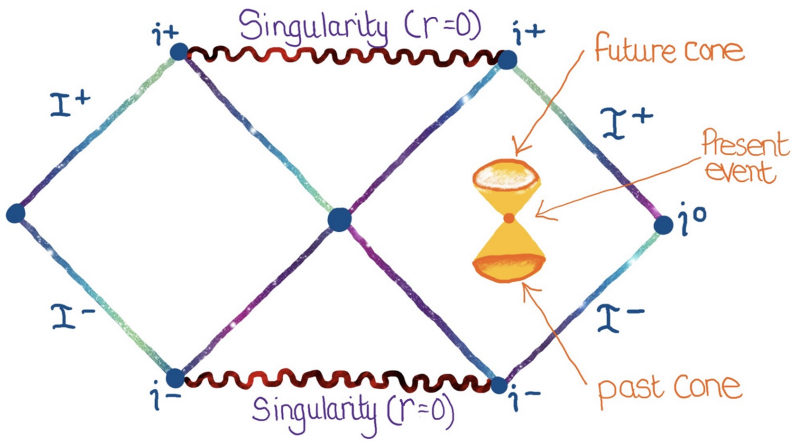

Figure 2: *A light cone somewhere in 'our' universe. In the diagram, time moves upward, with the past preceding the future.*

Saturday morning. Inside the cone above is your future, and inside the cone below is your past. The sides of the cones are actually the trajectories of beams of light through spacetime, coming out of the event. You'll notice that they are at the same angle as the boundaries of the box. This is what we call a light cone."

"Okay... And what about outside the light cone?" asked Lyra.

"It's just another part of the universe, however, to get to that part of the universe you would have to travel faster than the speed of light. Doing that, of course, cannot be done for objects with mass like you or I."

Lyra frowned.

"What would be an example of something outside the light cone?"

"Well, think of a nearby star system like Alpha Centauri. It's about four lightyears away or so, meaning it takes four years for a photon of light to reach it from Earth. That means that it takes at least four years for anything from Earth to have an effect on the star. If you, Lyra, wanted to throw a snowball at Alpha Centauri, you can't do it straight

away, you have to wait a long time for the snowball to eventually reach the system, even if you threw it at or close to the speed of light." [4]

"So because I can't do anything instantly, and can't travel faster than the speed of light, anything I do right now won't have an effect on the star for at least four years?" Lyra guessed.

"Exactly."

Orion pulled up his coat sleeve and looked at his wristwatch.

"At this moment, at thirty one minutes past ten on a Saturday morning," he announced. 'Anything you do, at this time, will have no effect whatsoever on Alpha Centauri. For the time being, you are what physicists say 'causally disconnected' from the star system. Something you do now may have some effect on the star in at least four years, but at this instant you are completely disconnected from it."

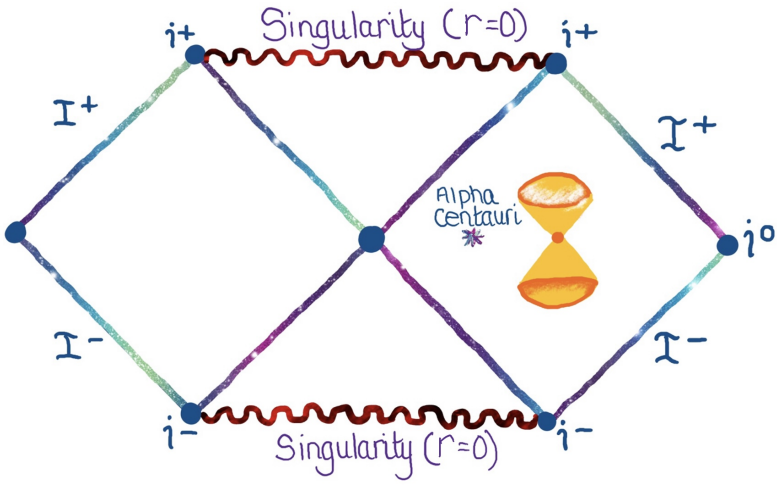

Figure 3: *Alpha Centauri's location in spacetime sits outside an observer's light cone, meaning it is causally disconnected at that point.*

Lyra thought it a little bleak that there were many parts of the universe which will never interact with one another.

"If I'm disconnected from things outside of the cone, I take it that means I'm connected to things inside the cone?" she wondered.

"Yes, you throwing a snowball at the star would be an event inside your light cone. And then the snowball hitting the star and evaporating would be in the light cone too."

"Is this why you said causal structure? Because the light cones contain my past and my future, but the diagram also shows things which are neither my past nor my future?"

Lyra's brain was beginning to hurt, but she could see why such a diagram would be of interest to physicists who are thinking about how time and space work.

"Couldn't have put it better myself," replied Orion. "Every possible event in your life, everyone you love, everyone you hate, every important moment, every happy day, every sad day, every breath you breathe, is contained inside this future cone. That doesn't mean all possible events will happen, however, it simply means that they could happen. You most likely will never throw a snowball at Alpha Centauri, but such an event could exist within your light cone, i.e. your future."

"It's almost like the speed of light determines whether you could be causally connected to something or not" said Lyra. She knew the speed of light was important, but not quite to this extent.

Orion grinned.

"You're almost correct. It's not the speed of light itself which is important, it's the fact it's constant for everyone, regardless. That's what gives us this picture, and allows us to think more deeply about the events we may or may not play a part in. It's why the sides of the light cones have a constant slope, and why everyone will observe the same constant slope." [5]

"But isn't the speed of light was related to relativity?" Lyra pointed out.

Orion laughed.

"Indeed! Relativity is somewhat misleading. The theory of relativity is actually predicated upon the invariance of the speed of light for all frames of reference, which is where all of this comes from."

"Okay. We have this Penrose diagram which apparently describes some infinite universe, and the box on the right has special boundaries," said Lyra. "And we have these light cone things that can tell us about the past and future. Where do black holes fit into all this?"

Although her mind was racing, Lyra's body was still and beginning to shiver in the cold. Orion now directed his stick to the upper triangle in the snow.

"There's your black hole." he said.

Orion then swept the stick along the right-hand boundary of the triangle.

"And there's the event horizon of the black hole, and here," said Orion, circling the horizontal line at the top of the diagram. "Is the singularity of the black hole."

"But the singularity indicates a breakdown right?" asked Lyra, recalling what her father had explained toward the beginning of their walk.

"Correct. It's actually thanks to Roger Penrose and Stephen Hawking that we know General Relativity predicts singularities." replied Orion. "I've detailed to you before how General Relativity is all about gravity arising as a result of the curvature of spacetime." [1]

"Yes, I remember" said Lyra, grimly. Of the numerous conversations with her father about mathematics and physics, the topic of General Relativity was most gruelling.

Orion ignored Lyra's distaste and continued.

"The stronger the gravity, the stronger the curvature of spacetime, and vice versa. For black holes, singularities arise when we see that there is infinite curvature in the centre. There was even a time when people thought there was a singularity at the event horizon."

"I thought the event horizon was the point of no return, does that have anything to do with singularities?" asked Lyra.

"Not as such. You see, the first mathematical solution to Einstein's equations for General Relativity was called the Schwarzschild solution, named after its discoverer, Karl Schwarzschild. Such an equa-

tion describes spacetime around a non-rotating and isolated source of mass, like a non-rotating black hole, for instance. However, this important equation was written in a coordinate system which blew up to infinity at the event horizon of the black hole, making people believe something had gone wrong there."

"What's so special about which coordinate system is being used?"

Orion looked out at the hill before them, leaning the stick against the trunk of the towering oak tree and tucking his hands into his pockets.

"Coordinates are very useful for writing down equations of physics, and in most cases we use coordinates which make those equations simpler to write down, or simpler to interpret. But sometimes coordinates aren't very good. Think of the Earth, much like this hill the Earth is curved, and we could use coordinates like latitude and longitude to describe a point on it. But what if I ask the question, 'What is above the north pole' for Earth? Suddenly, trying to use coordinates like latitude and longitude completely breaks down. Despite this, we know that there does exist a point above the north pole, somewhere in the atmosphere or even in space perhaps, but you would never know this if you only used latitude and longitude."

Lyra followed her father's gaze toward the hill, imagining trying to use latitude and longitude to locate a point on it. When she considered trying to do the same but for a point above peak of the slope, she couldn't do it, and then it clicked.

"So the coordinates for the Schwarz...-"

"Schwarzschild, yes" said Orion, helpfully.

"Are they a bit like using latitude and longitude to describe a point above the north pole, which is why they blow up to infinity?" Lyra asked gingerly. She began meandering in an uneven loop to keep warm amidst the freezing air.

"Exactly. And so physicists have proposed different coordinates where they didn't blow up at the event horizon of the black hole." said Orion. "Can you guess what the next question is?"

"Is the singularity in the centre of the black hole because of the coordinates?"

Lyra suspected the answer, which made her all the more mystified.

"No, it's not. There is a particular mathematical object which is of central importance in General Relativity. It is called the Riemann curvature tensor, and it essentially contains all the information related to how spacetime has curved in response to some mass."

Lyra was beginning to feel out of her depth now, but she continued listening intently all the same, hoping to catch some grain of understanding. Orion continued.

"The important thing about tensors is that they don't care about a coordinate system, they remain true regardless of the coordinate system. They don't care whether we use latitude and longitude, or any other type of coordinates. We can use the Riemann curvature tensor to calculate the curvature at the centre of the black hole. In particular, the Riemann tensor is used to calculate what is named the Kretschmann scalar, which is a single number telling us exactly how curved the spacetime actually is. When we do that for a Schwarzschild black hole, it diverges. That's how we know the singularity at the centre of a black hole isn't just a result of the coordinates we're using." [6]

Lyra took a reprieve from her loop walking, absorbing the information. Although lots of jargon was used, she could see there was some number that didn't care about coordinates, which sprung to infinity in the centre of a black hole. After a minute or two of looking once again at the Penrose diagram on the ground, she turned her head to her father.

"What about the other parts of this drawing? The other box looks like some kind of opposite universe or something..."

"Good guess!" exlaimed Orion. "Our universe has an opposite copy, and the black hole in the diagram also has an opposite copy, called a white hole."

"A white hole?" laughed Lyra, for it was such an overtly obvious name for a black hole's opposite.

"Yes." said Orion, not seeing the humour. "Black holes attract matter and energy, white holes emit it. White holes also have a singularity, emitting energy into the future, except that energy seemingly comes from nowhere. Physicists tend to ignore white holes for this reason."

"So we only care about our universe and the black hole, because they're physically relevant?" Lyra asked.

"Precisely. Although, to be frank, even the physically relevant parts of the diagram aren't hugely accurate. The black hole here is eternal for an observer in our universe, rather than being the result of a collapsing star. But as I said earlier, these diagrams are a good exposition of causal structure for spacetimes, which is why they're interesting to physicists." Orion explained.

"You mentioned earlier that Penrose showed General Relativity predicts singularities, did you just mean a black hole?" Lyra asked.

"No. All black holes have some kind of singularity in them, a rotating black hole even has a singularity in the shape of a ring! But Penrose showed that singularities arise naturally in General Relativity, and he did it using arguments from a branch of mathematics called topology. Topology relates to how geometric shapes deform and change, and he showed that no matter how one might deform spacetime in some way, certain situations lead to all possible paths to a singularity. That's what won him the Nobel prize" replied Orion. [7]

"What happens when you reach the singularity?" pondered Lyra, aloud.

"No one knows. And perhaps we will never know. For the singularity in a black hole, time and space appear to come to an end, and physics breaks down in those regions."

"You also mentioned Stephen Hawking as well though, what did he do?"

"Well, Penrose initially worked on black holes, and then Hawking essentially used Penrose's arguments and applied them to cosmology, to the universe at large," said Orion, picking up his stick again and pointing to the singularity in the diagram. "And he showed that just like a black hole, the universe has a singularity contained within it. Hawking showed that using General Relativity, and our models of cosmology, that the universe must have started from an initial singularity."

Lyra had heard something like this before, though didn't know it was because of Stephen Hawking. Something, however, was irking her.

"If singularities show that the theory is broken somehow, then how much of this is actually true?" she challenged.

Orion paused to think. It was an oddly complicated question.

"If I may, I think I can answer your question with another question: is any theory of physics ever actually true?" Orion asked, looking down at his daughter's befuddled face.

"I would hope so..." said Lyra, cautiously.

"Personally, I'm of the opinion that no theory of physics will ever be true, but rather a good approximation. There will always be things we don't know, science will never be finished. We do scientific research as though we are trying to find absolute truth, because if we aim for the stars we might just get the Moon. We do it because we may discover a rigorous explanation for things we can currently measure, which is the best we can do."

Lyra was silent, she almost looked a little disappointed. Orion continued.

"As to whether singularities really exist? Probably not, but the fact they appear motivates us to come up with a better theory. At this junction, it seems only some quantum theory of gravity could provide an explanation as to the nature of the centre of a black hole, or the beginning of our universe."

Lyra stared down at the patchwork of boxes and triangles below, her face forlorn. She understood the point her father was making, but still felt a little disheartened at the thought of never knowing the 'truth' about the universe, whatever that may be. Orion could see this and plodded through the snow towards her. He paused and crouched to his knees, directing the stick once again to the diagram.

"Do you want to know what my favourite thing about black holes is?" he said, looking up to his daughter.

Lyra didn't say anything at first, until she finally caved.

"Sure, tell me" she replied, the intrigue ebbing away at her disappointment.

Orion began drawing on the diagram again. This time he drew a light cone beyond the event horizon of the black hole, still directed vertically, just as before. The future directed cone pointed towards the singularity, and the past directed cone pointing out of the horizon. Orion looked back up to Lyra.

"What do you think you see when you fall into a black hole, do you think you can see the singularity?" he asked, mysteriously.

"I don't know, I guess not because it just sucks everything in, including light?" she answered, hesitantly.

"Maybe. Or maybe because of a much deeper reason. Look again at the diagram, where is the future light cone directed to?"

"It's pointing to the singularity..." said Lyra, still not understanding the point of the question.

"Exactly. That means the singularity is contained within the future light cone. So for a person falling into the black hole, past the event horizon, the singularity quite literally becomes a point in time. For us observers outside the black hole, it is a location in space at the centre of the black hole. But, for the unfortunate person falling in, it is now a part of their future."

Lyra was astounded. But then, just as she was processing this profound information, Orion said,

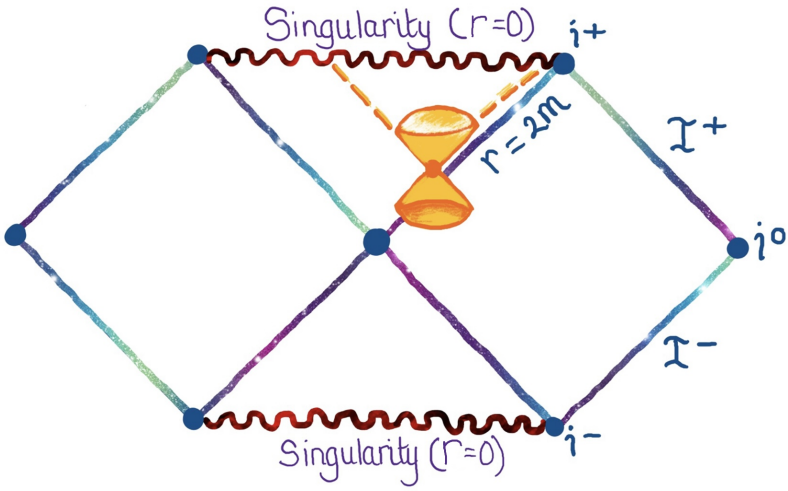

Figure 4: *The singularity of the black hole is contained within the future light cone of an observer who has fallen past the event horizon. Whether they like it or not, alive or dead, an observer is going to reach the singularity. It is their future.*

"And that's why you can never see the singularity. It's the same reason you can't see tomorrow, the same reason you can't see next week, the same reason you can't see all the amazing things that will one day happen to you. You can't see the singularity because it's in the future."

By this point, Lyra had goosebumps, and not from the cold. As Lyra was deep in thought, in awe, a frosty leaf came gently falling down upon the markings in the snow. In an odd way, the singularity being in the future made sense. After all, there is no getting out of a black hole once falling in, for at that point one's fate is literally sealed, destined to be pulled further and further inwards to the centre, unable to pull away. The leaf had landed inside the black hole on the diagram, before being carried away beyond the singularity by the wind. Lyra then turned back to Orion, who had been gazing at the Penrose diagram just like Lyra.

"I may not be able to see next week, Dad, but I can see us having some cake and tea at the cafe in the next half an hour" she quipped.

Orion laughed.

"You truly are a soothsayer! Come on then, let's go have some tea." he said, laying the stick down once again by the tree and guiding Lyra back to the path and up the hill. Looking back at the drawing left in the snow as they banked away, Lyra felt she knew less now than at the start.

"There might not be absolute truth for me to find, but I guess there's plenty enough mystery in the universe to discover anyhow." she thought.

.1 References

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